# PROBLEM SET 9

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**Problem 1.** *If $0 < p < 1$, let $\beta_p = p\delta_1 + (1-p)\delta_0$, and let $\beta_p^{(n)}$ be the $n$th convolution power of $\beta_p$,*

$$\beta_p^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \delta_k.$$

*If $a > 0$, let*

$$\lambda_a = e^{-a} \sum_{0}^{\infty} \frac{a^k}{k!} \delta_k.$$

*$\lambda_a$ is called the* Poisson distribution *with parameter $a$. Prove the following.*

*(1) The mean and variance of $\lambda_a$ are both $a$.*
*(2) $\lambda_a * \lambda_b = \lambda_{a+b}$.*
*(3) $\beta_{a/n}^{(n)}$ converges vaguely to $\lambda_a$ as $n \to \infty$.*

*Proof.* Suppose John is playing tossing coins with Jane, John wins a dollar from Jane if it's head, and nothing if it's tail (how unfair!). The coin is uneven, with probability $p$ showing head up in a single toss. Let $n$ be the number of tosses Jane made, and let $X_n$ be the money John earned from Jane in the $n$-th round. It is clear that $X_1, X_2, \dots$ are independent random variables, whose probability distribution is precisely $P_{X_n} = \beta_p$. Now let $S_n = X_1 + \cdots + X_n$ be the money John earned in total in the first $n$ rounds, we have

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

So $P_{S_n} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k$. On the other hand, $P_{S_n} = P_{X_1 + \cdots + X_n} = P_{X_1} * \dots P_{X_n} = \beta_p^{*n}$. By the end of this problem, we will show if $n$ is big enough and $p$ is small with $np = a$, then the distribution of $S_n$ is almost a Possion distribution $\lambda_a$.

(1) Let $Y$ be the random variable with $P_Y = \lambda_a$, *i.e.* $P(Y = k) = e^{-a} \frac{a^k}{k!}$. Then

$$\mathbf{E}(Y) = \sum_{k=0}^{\infty} k \cdot P(Y = k) = \sum_{k=1}^{\infty} e^{-a} \frac{a^k}{(k-1)!} = ae^{-a} \sum_{k=1}^{\infty} \frac{a^{k-1}}{(k-1)!} = a,$$

and

$$\begin{aligned}\sigma^2(Y) &= \sum_{k=0}^{\infty}(k-\mathbf{E}(Y))^2\cdot P(Y=k) = \sum_{k=0}^{\infty}(k^2-2ka+a^2)e^{-a}\frac{a^k}{k!}\\ &= e^{-a}\sum_{k=1}^{\infty}k^2\frac{a^k}{k!} - 2ae^{-a}\sum_{k=1}^{\infty}k\frac{a^k}{k!} + a^2e^{-a}\sum_{k=0}^{\infty}\frac{a^k}{k!}\\ &= e^{-a}[\sum_{k=2}^{\infty}k(k-1)\frac{a^k}{k!} + \sum_{k=1}^{\infty}k\frac{a^k}{k!}] - 2a^2 + a^2\\ &= e^{-a}[a^2e^a + ae^a] - a^2 = a.\end{aligned}$$

(2) Let $Z$ be a random variable with $P_Z = \lambda_b$ and $Y, Z$ independent. Then

$$\begin{aligned}P(Y+Z=k) &= \sum_{i=0}^{k}P(Y=i, Z=k-i) = \sum_{i=0}^{k}P(Y=i, Z=k-i)\\ &= \sum_{i=0}^{k}P(Y=i)P(Z=k-i) = \sum_{i=0}^{k}e^{-a}\frac{a^i}{i!}e^{-b}\frac{b^{k-i}}{(k-i)!}\\ &= e^{-(a+b)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}a^ib^{k-i} = e^{-(a+b)}\frac{(a+b)^k}{k!}.\end{aligned}$$

So $\lambda_a * \lambda_b = P_Y * P_Z = P_{Y+Z} = \lambda_{a+b}$.

(3) Let $Y_n$ be a random variable with $P_{Y_n} = \beta_{a/n}^{*n}$, *i.e.*

$$P(Y_n = k) = \binom{n}{k}(\frac{a}{n})^k(1-\frac{a}{n})^{n-k}$$

for $k = 0, 1, \ldots, n$. Taking limit $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty}P(Y_n=k) &= \lim_{n\to\infty}\binom{n}{k}(\frac{a}{n})^k(1-\frac{a}{n})^{n-k}\\ &= \lim_{n\to\infty}\frac{n!}{n^k\cdot(n-k)!}\frac{a^k}{k!}(1-\frac{a}{n})^n(1-\frac{a}{n})^{-k}\\ &= e^{-a}\frac{a^k}{k!}\lim_{n\to\infty}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})\\ &= e^{-a}\frac{a^k}{k!}.\end{aligned}$$

It follows that $P(Y_n \le k)$ converges to $P(Y \le k)$ as $n \to \infty$ for all $k$, and thus $P(Y_n \le x) \to P(Y \le x)$ as $n \to \infty$ for all $x \in \mathbb{R}$ as $Y_n, Y$ are discrete. Hence by Prop. 7.19, $\beta_{a/n}^{*n} = P_{Y_n} \to P_Y = \lambda_a$ vaguely as $n \to \infty$.

□

*Remark.* We note that the appearance of random variables is unnecessary in solving problem 1, one can simply use the corresponding definitions for distributions. Also, the existence of sample spaces and such random variables is not trivial. However I find it helps me understand the problem better by putting random variables in the picture.

**Problem 2.** *If* $\sum_1^\infty n^{-2}\sigma_n^2 < \infty$*, then* $\lim n^{-2}\sum_1^n \sigma_j^2 = 0$*. If* $\{a_n\} \subset \mathbb{C}$ *and* $\lim a_n = a$*, then* $\lim n^{-1}\sum_1^n a_j = a$*.*

*Proof.* (1) We remark this is a special case of Kronecker's lemma. Let $S_j = \sum_{k=1}^{j} k^{-2}\sigma_k^2$, and $s = \lim S_j$. Using summation by part we get

$$\frac{1}{n^2}\sum_{j=1}^{n}\sigma_j^2 = S_n - \frac{1}{n^2}\sum_{j=1}^{n-1}[(j+1)^2 - j^2]S_j.$$

Now pick any $\epsilon > 0$ and choose $N$ so that $S_j$ is $\epsilon$ close to $s$ for $n > N$. Then

$$\begin{aligned} &S_n - \frac{1}{n^2}\sum_{j=1}^{n-1}[(j+1)^2 - j^2]S_j \\ =&S_n - \frac{1}{n^2}\sum_{j=1}^{N-1}[(j+1)^2 - j^2]S_j - \frac{1}{n^2}\sum_{j=N}^{n-1}[(j+1)^2 - j^2]S_j. \end{aligned}$$

As $n \to \infty$, the first term converges to $s$, the second goes to 0. For the third term we have

$$\begin{aligned} &-\frac{1}{n^2}\sum_{j=N}^{n-1}[(j+1)^2 - j^2]S_j \\ =&-\frac{1}{n^2}\sum_{j=N}^{n-1}[(j+1)^2 - j^2]s - \frac{1}{n^2}\sum_{j=N}^{n-1}[(j+1)^2 - j^2](S_j - s) \\ =&-\frac{n^2 - N^2}{n^2}s - -\frac{1}{n^2}\sum_{j=N}^{n-1}[(j+1)^2 - j^2](S_j - s). \end{aligned}$$

Here the first term goes to $-s$, and the second is bounded by $\epsilon\frac{n^2-N^2}{n^2} \le \epsilon$. This proves

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{j=1}^{n}\sigma_j^2 \le \epsilon$$

for all $\epsilon > 0$, therefore the limit is 0.

(2) Since $\{a_n\}$ is convergent, it is bounded by some $M > 0$. Fix $\epsilon > 0$, there is $N$ so that for all $n > N$ we have $|a_n - a| \le \epsilon$. Thus for $n > N$ we have

$$\frac{1}{n}\sum_{1}^{n} a_j = \frac{1}{n}\sum_{1}^{N} a_j + \frac{1}{n}\sum_{N+1}^{n} a_j.$$

Note that the first term $\frac{1}{n}\sum_1^N a_j$ is bounded by $NM/n \to 0$ as $n \to \infty$. As for the second term, we have

$$|\frac{1}{n}\sum_{N+1}^{n} a_j - a| \le |\frac{1}{n}\sum_{N+1}^{n}(a_j - a)| + \frac{N}{n}|a| \le \epsilon + \frac{N}{n}|a|.$$

Hence $\limsup|\frac{1}{n}\sum_1^n a_j - a| \le \epsilon$ for all $\epsilon > 0$. This proves $\lim\frac{1}{n}\sum_1^n a_j = a$. □

**Problem 3.** *If $\{X_n\}$ is a sequence of independent random variables such that $\mathbf{E}(X_n) = 0$ and $\sum_1^\infty \sigma^2(X_n) < \infty$, then $\sum_1^\infty X_n$ converges almost surely.*

*Proof.* We denote $A_{k,n}(\epsilon) = \{\omega \in \Omega : \max_{0\le j\le n} |X_k + X_{k+1} + \cdots + X_{k+j}| \ge \epsilon\}$ and $A_k(\epsilon) = \cup_{n=0}^{\infty} A_{k,n}(\epsilon)$, $A(\epsilon) = \cap_k A_k(\epsilon)$. It is clear $A_{k,n} \subset A_{k,n+1}$ and $A_k \supset A_{k+1}$. Notice by Kolmogorov inequality, we have

$$P(A_{k,n}(\epsilon)) \le \epsilon^{-2} \sum_{j=k}^{n+k} \sigma^2(X_j) \le \epsilon^{-2} \sum_{j\ge k} \sigma^2(X_j).$$

Hence $P(A_k(\epsilon)) = P(\cup_n A_{k,n}(\epsilon)) = \lim_n P(A_{k,n}(\epsilon)) \le \epsilon^{-2} \sum_{j\ge k} \sigma^2(X_j)$. So

$$P(A(\epsilon)) = P(\cap_k A_k(\epsilon)) = \lim_{k\to\infty} P(A_k(\epsilon)) = 0.$$

Now we claim $\sum_1^\infty X_n$ is Cauchy a.s.. Indeed, suppose $\omega \in \Omega$ so that $\sum_1^\infty X_n$ is not Cauchy, then there exists $\epsilon > 0$, so that for all $N > 0$, we can find $k > N$ and $n > 0$ such that $|X_k(\omega) + \cdots + X_{k+n}(\omega)| \ge \epsilon$. This means we may find an increasing sequence $k_1, k_2, \cdots \to \infty$ and $n_1, n_2, \ldots$ so that $|X_{k_1}(\omega) + \cdots + X_{k_1+n_1}(\omega)| \ge \epsilon$. Thus $\omega \in A_{k_j,n_j}(\epsilon) \subset A_{k_j}(\epsilon)$ for all $j$. Hence $\omega \in \cap_j A_{k_j}(\epsilon) = A(\epsilon)$, which is a measure zero set. And the set where $\sum_1^\infty X_n$ fails to be Cauchy is contained in $\cup_{\epsilon>0} A(\epsilon) = \cup_{m\ge 1} A(\frac{1}{m})$, which is a measure zero set. □

**Problem 4.** *If $\{X_n\}$ is a sequence of i.i.d. random variables which are not in $L^1$, then $\limsup n^{-1}|\sum_1^n X_j| = \infty$ almost surely.*

*Proof.* Let $S_n = \sum_1^n X_j$, we notice that $|X_n| = |S_n - S_{n-1}| \le |S_n| + |S_{n-1}|$, so $\limsup n^{-1}|S_n| \ge \frac{1}{2} \limsup n^{-1}|X_n|$. Therefore it suffices to show $\limsup |X_n|/n = \infty$ a.s., or equivalently $\limsup |X_n|/n \ge m$ a.s. for all $m \in \mathbb{N}$.

Now we fix $m \in \mathbb{N}$, let $A_n = \{|X_n| \ge mn\}$. We claim $\sum_1^\infty P(A_n) = \infty$. Indeed

$$\begin{aligned}
\sum_1^\infty P(A_n) &= \sum_1^\infty \lambda(\{t : |t| > mn\}) = \sum_{n=1}^\infty \sum_{k=n}^\infty \lambda(\{t : mk < |t| \le mk + m\}) \\
&= \sum_{k=1}^\infty k\lambda(\{t : mk < |t| \le mk + m\}) \\
&= \frac{1}{m} \sum_{k=1}^\infty (mk + m)\lambda(\{t : mk < |t| \le mk + m\}) \frac{k}{k+1} \\
&\ge \frac{1}{2m} \int |t| d\lambda(t) = \infty.
\end{aligned}$$

Since $X_n$'s are independent, so are $A_n$'s. Therefore by Borel-Cantelli lemma

$$P(\limsup A_n) = 1.$$

This precisely means $\limsup |X_n|/n \ge m$ almost surely. □

**Problem 5.** *(Shannon's Theorem) Let $\{X_i\}$ be a sequence of independent random variables on the sample space $\Omega$ having the common distribution $\lambda = \sum_1^r p_j \delta_j$ where $0 < p_j < 1$, $\sum_1^r p_j = 1$, and $\delta_j$ is the point mass at $j$. Define random variables $Y_1, Y_2, \ldots$ on $\Omega$ by*

$$Y_n(\omega) = P\left(\{\omega' : X_j(\omega') = X_j(\omega) \text{ for } 1 \le j \le n\}\right).$$

*Prove the following.*

1. *$Y_n = \prod_1^n p_{X_j}$.*
2. *$n^{-1} \log Y_n \to \sum_1^r p_j \log p_j$ almost surely.*

*Proof.* (1) By independency, $Y_n(\omega) = \prod_1^n P(X_j = X_j(\omega)) = \prod_1^n p_{X_j(\omega)}$. The last equality can be a little confusing. We explain as follows. Denote $X_j(\omega) = k$, then since $X_j$ has distribution $\lambda = \sum_i p_i \delta_i$, we have $P(X_j = k) = p_k$, *i.e.* $P(X_j = X_j(\omega)) = p_{X_j(\omega)}$.

(2) From (1), we have $n^{-1} \log Y_n = n^{-1} \sum_1^n \log p_{X_j}$. Since $X_j' s$ are independent and identically distributed, so are $\log p_{X_j}$'s whose expectation is

$$\mathbf{E} = \sum_{k=1}^{r} P(X_j = k) \cdot \log p_k = \sum_{1}^{r} p_k \log p_k.$$

Therefore by central limit theorem, $n^{-1} \log Y_n \to \sum_1^r p_k \log p_k$ almost surely.

□